where D_a is the ambipolar diffusion coefficient and $m = C_e = C_i$.[4] Since $D_i \ll D_a$, the last term is negligible; the diffusion is still ambipolar, reduced by the magnetic field.

Summarizing, from order-of-magnitude arguments, it is concluded that if the probe current is limited by the ambipolar region, it is reduced by the applied magnetic field. If it is limited by the mobility region, it may have values close to those of the nonmagnetic case. The transition from one mode to the other depends certainly on the thickness of the viscous boundary layer, in a way fitting to the results of Wilson and Turcott. The critical current density certainly will be reduced when the magnetic field is strengthened as it is shown by their results.

One must have noticed that only order-of-magnitude results have been presented. However, all these indicate that in analyzing a performance of an electrostatic probe in an applied magnetic field, one has to include both its viscous and magnetogasdynamic boundary layers.

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Resolution of the 90° Euler Angle Singularity

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Introduction

A WELL-KNOWN peculiarity in the Euler angle representation of the attitude of a rigid body is the apparent infinity in the derivatives of the first and third angles when the second angle is 90°. Since there is no real physical difficulty at this point, it should be possible to resolve the mathematical difficulties. In this note we will develop a set of derivative equations for the 90° region that is singularity free.

Analysis

Let xyz denote a set of Cartesian axes imbedded in a rigid body, and let pqr denote the components along xyz of the body's angular velocity vector with respect to space. The attitude of xyz with respect to space is specified in terms of an Euler angle set α, β, and γ taken in that order around successive positions of x, y, and z. In the zero position, $\alpha = \beta = \gamma = 0$, xyz is coincident with some spatial frame XYZ. The problem is the usual one of determining α, β, and γ given p, q, and r as prescribed functions of time.†

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† In the classical Euler angle scheme the first and third rotations are taken around the same body axis,[1] whereas here one rotation is taken around each body axis. However, the present scheme is used more widely in engineering analysis, and the angles are still usually referred to as Euler angles. We are following this convention.

The equations relating p, q, r and the Euler angle derivatives are

$$
\begin{aligned}
p &= \dot\alpha \cos\beta \cos\gamma + \dot\beta \sin\gamma \\
q &= -\dot\alpha \cos\beta \sin\gamma + \dot\beta \cos\gamma \\
r &= \dot\alpha \sin\beta + \dot\gamma
\end{aligned}
\tag{1}
$$

Solving for the derivatives

$$\dot\alpha = (1/\cos\beta)(p \cos\gamma - q \sin\gamma) \tag{2a}$$

$$\dot\beta = p \sin\gamma + q \cos\gamma \tag{2b}$$

$$\dot\gamma = r - (\sin\beta/\cos\beta)(p \cos\gamma - q \sin\gamma) \tag{2c}$$

Equations (2) show that $\dot\alpha$ and $\dot\gamma$ are apparently infinite when $\beta = 90°$, although $\dot\beta$ is well-defined. We now show that $\dot\alpha$ and $\dot\gamma$ are, in fact, not infinite but indeterminate, and then we obtain their limiting values as $\beta \to 90°$.

In Eqs. (1) set $\beta = 90°$. This yields

$$p = \dot\beta \sin\gamma \qquad q = \dot\beta \cos\gamma \qquad r = \dot\alpha + \dot\gamma \tag{3}$$

Combining the first two of Eqs. (3) yields

$$p \cos\gamma - q \sin\gamma = 0 \tag{4}$$

Equation (4) is the key relation for it shows that the numerators, as well as the denominators in Eqs. (2a) and (2c) are zero at $\beta = 90°$, so that $\dot\alpha$ and $\dot\gamma$ are indeterminate. Physically, the indeterminancy exists since the α and γ rotation axes are in spatial coincidence, and $\dot\alpha$ and $\dot\gamma$ cannot be separated; all that is known, as shown by the last of Eqs. (3), is that their sum is r.

To obtain the limiting values as $\beta \to 90°$, apply L'Hospital's rule to Eqs. (2a) and (2c). Since $d/d\beta = (1/\dot\beta)(d/dt)$, $(1/\dot\beta)$ cancels when taking the derivatives of numerators and denominators, and we need only take d/dt. The result is

$$
\begin{aligned}
&\lim_{\beta \to 90°}[(p \cos\gamma - q \sin\gamma)/\cos\beta] = \\
&\quad \lim_{\beta \to 90°}[(\sin\beta/\cos\beta)(p \cos\gamma - q \sin\gamma)] = \\
&-(1/\dot\beta)[(\dot p \cos\gamma - \dot q \sin\gamma) - (p \sin\gamma + q \cos\gamma)\dot\gamma] = \\
&\quad (1/\dot\beta)[-\dot p \cos\gamma + \dot q \sin\gamma + \dot\beta\dot\gamma]
\end{aligned}
\tag{5}
$$

upon noting the expression for $\dot\beta$ from Eq. (2b).

Equation (5) shows that the limits which were taken to determine $\dot\alpha$ and $\dot\gamma$ depend, in fact, on $\dot\gamma$ itself. Now substitute Eq. (5) into Eq. (2c) and obtain

$$\dot\gamma = r + (1/\dot\beta)(\dot p \cos\gamma - \dot q \sin\gamma - \dot\beta\dot\gamma)$$

and solving for $\dot\gamma$

$$\dot\gamma = r/2 + (1/2\dot\beta)(\dot p \cos\gamma - \dot q \sin\gamma) \tag{6a}$$

Now substitute for $\dot\gamma$ into Eq. (5) and the result into Eq. (2a) to obtain $\dot\alpha$

$$\dot\alpha = r/2 + (1/2\dot\beta)(-\dot p \cos\gamma + \dot q \sin\gamma) \tag{6b}$$

Equations (6a) and (6b) give the limiting values of $\dot\alpha$ and $\dot\gamma$ as $\beta \to 90°$. Simultaneous with these results, from Eq. (4), $\tan\gamma = p/q$ so that

$$\sin\gamma = \pm p/(p^2 + q^2)^{1/2} \qquad \cos\gamma = \pm q/(p^2 + q^2)^{1/2} \tag{7}$$

Substituting Eqs. (7) into Eq. (2b)

$$\dot\beta = \pm (p^2 + q^2)^{1/2} \tag{8}$$

and then combining Eqs. (7) and (8) into Eqs. (6a) and (6b) yields

$$
\begin{aligned}
\dot\alpha &= \tfrac{1}{2}[r - (\dot p q - \dot q p)/(p^2 + q^2)] \\
\dot\gamma &= \tfrac{1}{2}[r + (\dot p q - \dot q p)/(p^2 + q^2)]
\end{aligned}
\tag{9}
$$

We now summarize our results. Whenever β is within some ϵ neighborhood of 90° ($\epsilon \to 0$), $\dot{\alpha}$ and $\dot{\gamma}$ are obtained from Eqs. (6a) and (6b) or Eqs. (9), and $\dot{\beta}$ is obtained from Eq. (2b) or Eq. (8), where in the latter equation the $\pm$ signs must be distinguished according to whether β has been increasing or decreasing in its passage through 90°. Outside of this ϵ neighborhood $\dot{\alpha}\dot{\beta}\dot{\gamma}$ are always given by Eqs. (2).

Reference

[1] Whittaker, E. T., *Analytical Dynamics* (Dover Publications, New York, 1944), 4th ed., pp. 9 and 10.

Linearized Impulsive Guidance Laws

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I. Introduction

An impulsive guidance law defines an impulsive correction, which will nullify the effects of position and velocity errors along a trajectory while satisfying specified mission constraints. Examples are fixed and variable arrival time guidance laws.[1] By starting with a generalized linear expression for the constraints, a general solution can be obtained to the impulsive guidance problem.

The specification of the nominal trajectory and the initial position perturbation leaves three conditions to be determined to define a new trajectory and one condition to determine the terminal point. There are thus four constraints that must be imposed (coplanar problems leave only three constraints to be imposed). Two situations will be considered: the case of four independent constraints and the case of minimizing a specified function while satisfying three independent constraints.

II. Four Independent Constraints

Four independent scalar constraints can be imposed which are functions of the perturbations in the initial (subscript i) and final (no subscript) state vectors and in the time of flight (perturbations in other variables can be converted to perturbations in time of flight).

$$A\delta R_i + B\delta V_i + C\delta R + D\delta V + E\delta t = [0] \quad (1)$$

where A, B, C, and D are 4×3 guidance constraint matrices, E is a 4×1 guidance constraint vector, [0] is a null vector, and δV_i is the velocity error after the correction δV_c has been applied ($\delta V_c = \delta V_i - \delta V_{initial}$). The final state error is related to the initial state error through the (partitioned) time transition matrix P.†

$$\left.\begin{aligned} \delta R &= P_1\delta R_i + P_2\delta V_i + V_r\delta t \\ \delta V &= P_3\delta R_i + P_4\delta V_i + A_r\delta t \end{aligned}\right\} \quad (2)$$

where V_r and A_r are the differences in the velocity and acceleration between the nominal and target trajectories at the terminal point. If there is no target trajectory, V_r and A_r become the velocity V_n and acceleration A_n of the nominal trajectory. Inserting Eq. (2) into Eq. (1) results in

$$[A + CP_1 + DP_3]\delta R_i + [B + CP_2 + DP_4]\delta V_i + [CV_r + DA_r + E]\delta t = [0] \quad (3)$$

Solving for δV_i and δt,

$$\begin{bmatrix} \delta V_i \\ \delta t \end{bmatrix} = N^{-1}M\delta R_i \quad (4)$$

where N and M are 4×4 and 4×3 matrices equal to

$$\left.\begin{aligned} N &= [B + CP_2 + DP_4\; CV_r + DA_r + E] \\ M &= [A + CP_1 + DP_3] \end{aligned}\right\} \quad (5)$$

Partitioning $N^{-1}M$ into a 3×3 matrix F and a 1×3 row vector G results in the fundamental guidance matrix and the corresponding time guidance vector

$$\delta V_i = F\delta R_i \qquad \delta t = G\delta R_i \quad (6)$$

where $N^{-1}M = \begin{bmatrix} F \\ G \end{bmatrix}$

III. Three Independent Constraints and a Function to be Minimized

In some cases it is desirable to minimize a quantity Q while satisfying three independent scalar constraints. The quantity Q can be generally expressed

$$Q = f(\delta V_i, \delta R, \delta V, \delta t) \quad (7)$$

The perturbations in δR and δV can be expressed as functions of δV_i and δt using Eq. (2).

The three independent constraints can be expressed as in Eq. (1) and combined with Eq. (2) to obtain [see Eq. (3)]

$$\left.\begin{aligned} H\delta R_i + K\delta V_i + L\delta t &= [0] \\ \delta V_i &= -K^{-1}H\delta R_i - K^{-1}L\delta t \end{aligned}\right\} \quad (8)$$

where H and K are 3×3 matrices, and L is a column vector. Upon substituting Eq. (8) into Eq. (7), Q is obtained as a function of δt. The derivative of Q with respect to δt defines the value of δt which minimizes Q. If this derivative fails to yield a value for δt, no linear solution exists.

Frequently, the magnitude of a vector is to be minimized, and the solution is known to exist. In this case, it is more convenient to differentiate the quantity before carrying out the previously mentioned substitutions. Differentiating Q with respect to δt results in

$$\omega = S[d(\delta V_i)/d(\delta t)] + T[d(\delta R)/d(\delta t)] + U[d(\delta V)/d(\delta t)] \quad (9)$$

where S, T, and U are row vectors, and ω is a scalar.

Eq. (8) can be differentiated to obtain

$$d(\delta V_i)/d(\delta t) = -K^{-1}L \quad (10)$$

Using Eq. (2)

$$\left.\begin{aligned} d(\delta R)/d(\delta t) &= P_2[d(\delta V_i)/d(\delta t)] + V_r \\ d(\delta V)/d(\delta t) &= P_4[d(\delta V_i)/d(\delta t)] + A_r \end{aligned}\right\} \quad (11)$$

Combining Eqs. (9–11)

$$\omega = -SK^{-1}L + T(-P_2K^{-1}L + V_r) + U(-P_4K^{-1}L + A_r) \quad (12)$$

The preceding equation can be directly solved for δt, which can then be inserted into Eq. (9).

IV. Examples of Constraints

The following examples of independent constraints lead to the coefficients in Eq. (1) by direct comparison:

1) Terminal position velocity (representing three independent constraints)

$$\delta R = 0 \quad (13)$$

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† The use of δt as a variable and the corresponding use of the time transition matrix is arbitrary. For example, range angle and the corresponding angle transition matrix could have been used.